

Hasse principle for character group of finitely generated field over the rational number field

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Abstract

In this paper, we show the Hasse principle for the character group of a finitely generated field over the rational number field. By applying this result, we obtain an algebraic proof of unramified class field theory of arithmetical schemes.

1 Introduction

The Hasse principle for the character group of a global field is known as a classical result ([2, p.180, 8.8 Corollary]). The classical class field theory is used to prove this result, especially the first inequality ([2, p.179, 8.4 Consequence]). The objective of this paper is to show the Hasse principle for the character group of a finitely generated field over the rational number field \mathbf{Q} , that is, to show the following results.

THEOREM 1.1. (Theorem 4.1)

- (i). Let k be a finitely generated field over \mathbf{Q} ; X , a regular algebraic curve over k ; and $R(X)$, the function field of X . Let $\widetilde{R(X)}_{\mathfrak{p}}$ be the function field of the Henselization of $\mathcal{O}_{X,\mathfrak{p}}$ for $\mathfrak{p} \in X \setminus (0)$. Then, the local-global map of the character groups

$$H^1(R(X), \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{\mathfrak{p} \in X \setminus (0)} H^1(\widetilde{R(X)}_{\mathfrak{p}}, \mathbf{Q}/\mathbf{Z})$$

is injective.

- (ii). Let the Kronecker dimension of k be n . For a certain set of n -dimensional local fields $\{k_{n,i}\}_{i \in P_n}$, the local-global map

$$H^1(k, \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{i \in P_n} H^1(k_{n,i}, \mathbf{Q}/\mathbf{Z})$$

is injective. Moreover,

$$H^1(R(X), \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{i \in P_n} H^1(R(X \times k_{n,i}), \mathbf{Q}/\mathbf{Z})$$

is injective.

By applying the above results, we obtain the following:

COROLLARY 1.1. (Corollary 4.1) Let \mathcal{X} be a connected normal scheme of finite type over $\text{Spec}(\mathbf{Z})$ where the characteristic of \mathcal{X} is 0. Suppose that $i \leq \dim(\mathcal{X})$. Then,

$$H^1(\mathcal{X}, \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{\mathfrak{p} \in \mathcal{X}_{(i)}} H^1(\kappa(\mathfrak{p}), \mathbf{Q}/\mathbf{Z})$$

is injective.

This result yields an algebraic proof of the unramified class field theory of arithmetical schemes ([10, p.270, Theorem (5.10)]), which is proved by using Chebotarev's density theorem in [10].

2 Notation

For a scheme X , $X_{(i)}$ is the set of points of codimension i , and $X^{(i)}$ is the set of points of dimension i . We denote by $\pi_1^{ab}(X)$ the abelian fundamental group of X . For an integer $m > 0$, we identify the étale cohomology group $H^1(X, \mathbf{Z}/m\mathbf{Z})$ with the group of all continuous homomorphisms $\pi_1^{ab}(X) \rightarrow \mathbf{Z}/m\mathbf{Z}$.

For an integral scheme X and $\mathfrak{p} \in X$, let $\kappa(\mathfrak{p})$ be the residue field at \mathfrak{p} ; $R(X)$, the function field of X ; $\mathcal{O}_{X,\mathfrak{p}}$, the local ring at \mathfrak{p} ; $\widehat{\mathcal{O}_{X,\mathfrak{p}}}$, the Henselization of $\mathcal{O}_{X,\mathfrak{p}}$; $\widehat{R(X)}_{\mathfrak{p}}$, its quotient field; $\widehat{\mathcal{O}_{X,\mathfrak{p}}}$, the completion of $\mathcal{O}_{X,\mathfrak{p}}$; $\widehat{R(X)}_{\mathfrak{p}}$, its quotient field; $\mathcal{O}_{X,\bar{\mathfrak{p}}}$, the strict Henselization of $\mathcal{O}_{X,\mathfrak{p}}$; and $R(X)_{\bar{\mathfrak{p}}}$, its quotient field. When an integer m is invertible in \mathcal{O}_X , μ_m denotes the sheaf of m -th roots of unity on the étale site X_{et} of X . For a local ring A , $K_n^M(A)$ denotes the Milnor K -group of degree n for A .

3 Higher local class field theory and its application

In this section, our objective is to prove Proposition 3.3, which is a generalization of [5, p.524, Lemma 5.4]; it is required to prove Theorem 4.1.

3.1 Higher local field theory

In this subsection, we review higher local field theory, which plays an important role in the proof of the main result.

Definition 3.1. A field K is called an *n-dimensional local field* if there is a sequence of fields k_n, \dots, k_0 satisfying the following conditions: k_0 is a finite field, k_i is a *Henselian* discrete valuation field with residue field k_{i-1} for $i = 1, 2, \dots, n$, and $k_n = K$.

For any field k of characteristic 0, and $r \geq 1$, let

$$H^r(k) = \varinjlim H^r(k, \mu_m^{\otimes(r-1)})$$

be the Galois cohomology groups.

Then, there exists a canonical isomorphism

$$\eta : H^{n+1}(K) \simeq \mathbf{Q}/\mathbf{Z}.$$

For $0 \leq r \leq n+1$, the canonical pairing

$$\langle \cdot, \cdot \rangle : H^r(K) \times K_{n+1-r}^M(K) \rightarrow H^{n+1}(K)$$

induces a homomorphism

$$\Phi_K^r : H^r(K) \rightarrow \text{Hom}(K_{n+1-r}^M(K), \mathbf{Q}/\mathbf{Z})$$

THEOREM 3.2. [6, 8] Let K be an n -dimensional local field and F be the $n-1$ -dimensional local field that is the residue field of K . Then,

(i). The correspondence

$$L \rightarrow N_{L/K} K_n^M(L)$$

is a bijection from the set of all finite abelian extensions of K to the set of all open subgroups of $K_n^M(K)$ of finite index.

(ii). Φ_K^r induces an isomorphism between $H^r(K)$ and the group of all continuous characters of finite order of $K_{n+1-r}^M(K)$ when $0 \leq r \leq n+1$.

(iii). We have the commutative diagram

$$\begin{array}{ccc} K_n^M(K) & \longrightarrow & \text{Gal}(K^{ab}/K) \\ \partial \downarrow & & \downarrow \\ K_{n-1}^M(F) & \longrightarrow & \text{Gal}(F^{ab}/F), \end{array}$$

where the horizontal arrows come from the class field theory, and the left vertical arrow ∂ is the boundary homomorphism.

In particular, an element $\chi \in H^1(K, \mathbf{Q}/\mathbf{Z})$ is unramified, that is, the corresponding cyclic extension of K is unramified, if and only if $\Phi_K^1(\chi)$ is trivial on $\text{Ker}(\partial)$.

3.2 Class field theory of schemes over Henselian discrete valuation field

Let Z be an excellent scheme. For $x \in Z^{(i)}$ and $y \in Z^{(i+1)}$, let

$$\partial_x^y : K_{*+1}^M(\kappa(y)) \rightarrow K_*^M(\kappa(x))$$

be the following homomorphism. Let W be the normalization of the reduced scheme \widehat{y} , and define ∂_x^y as

$$\partial_x^y = \sum_{x'} N_{\kappa(x')/\kappa(x)} \circ \partial_{x'}.$$

Here, x' ranges over all points of W lying over x , $\partial_{x'}$ denotes the tame symbol

$$K_{*+1}^M(\kappa(y)) \rightarrow K_*^M(\kappa(x'))$$

associated with the discrete valuation ring $\mathcal{O}_{Y,x'}$ and $N_{\kappa(x')/\kappa(x)}$ is the norm map

$$K_*^M(\kappa(x')) \rightarrow K_*^M(\kappa(x)).$$

Then,

$$\cdots \rightarrow \bigoplus_{x \in Z^{(2)}} K_{n+2}^M(\kappa(x)) \rightarrow \bigoplus_{x \in Z^{(1)}} K_{n+1}^M(\kappa(x)) \rightarrow \bigoplus_{x \in Z^{(0)}} K_n^M(\kappa(x))$$

is complex by [7]. For an integer n , we define

$$\mathrm{SK}_n(Z) = \mathrm{Coker} \left(\partial : \bigoplus_{y \in Z^{(1)}} K_{n+1}^M(\kappa(y)) \rightarrow \bigoplus_{x \in Z^{(0)}} K_n^M(\kappa(x)) \right).$$

We consider schemes \mathfrak{X} , X , and Y which satisfy the following assumption.

ASSUMPTION 3.3. \mathcal{O}_k : a Henselian discrete valuation ring with residue field F and quotient field k ,

\mathfrak{X} : a connected regular proper scheme over $S = \mathrm{Spec}(\mathcal{O}_k)$,

$X = \mathfrak{X} \otimes_{\mathcal{O}_k} k$ and $Y = \mathfrak{X} \otimes_{\mathcal{O}_k} F$.

Then, $X^{(1)} \subset \mathfrak{X}^{(2)}$, $\mathfrak{X}^{(1)} = X^{(0)} \cup Y^{(1)}$, and $\mathfrak{X}^{(0)} = Y^{(0)}$. Hence, we have the anti-commutative diagram

$$\begin{array}{ccccccc} \bigoplus_{v \in X^{(1)}} K_{n+1}^M(\kappa(v)) & \longrightarrow & \bigoplus_{u \in X^{(0)}} K_n^M(\kappa(u)) & \longrightarrow & \mathrm{SK}_n(X) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ \bigoplus_{y \in Y^{(1)}} K_n^M(\kappa(y)) & \longrightarrow & \bigoplus_{x \in Y^{(0)}} K_{n-1}^M(\kappa(x)) & \longrightarrow & \mathrm{SK}_{n-1}(Y) & \longrightarrow & 0. \end{array} \quad (1)$$

Therefore, we obtain the homomorphism

$$\partial : \mathrm{SK}_n(X) \rightarrow \mathrm{SK}_{n-1}(Y).$$

Let k be an n -dimensional local field. Suppose that X is a smooth proper algebraic curve over k . Then, we construct a homomorphism

$$\tau : \mathrm{SK}_n(X) \rightarrow \pi_1^{ab}(X).$$

Let K be the function field of X , $P = X^{(0)}$; $R_{\mathfrak{p}}$, the Henselization of $\mathcal{O}_{X,\mathfrak{p}}$ for $\mathfrak{p} \in P$; and $K_{\mathfrak{p}}$, the quotient field of $R_{\mathfrak{p}}$. Then, we define

$$I_K = \prod'_{\mathfrak{p} \in P} K_{n+1}^M(K_{\mathfrak{p}}),$$

where \prod' denotes the restricted product with respect to the subgroups $\text{Ker}(\partial_{\mathfrak{p}}) = K_{n+1}^M(R_{\mathfrak{p}})$ for $\mathfrak{p} \in P$.

Let χ be an element of $H^1(K, \mathbf{Q}/\mathbf{Z})$ and $\chi_{\mathfrak{p}}$ be the restriction of χ to $H^1(K_{\mathfrak{p}}, \mathbf{Q}/\mathbf{Z})$. Note that $\chi_{\mathfrak{p}}$ is unramified for almost all $\mathfrak{p} \in P$. Hence, if $a = (a_{\mathfrak{p}})_{\mathfrak{p} \in P}$ is an element of I_K , by Theorem 3.1 (iii) and the definition of I_K , we have

$$\langle \chi_{\mathfrak{p}}, a_{\mathfrak{p}} \rangle_{\mathfrak{p}} = \Phi_{K_{\mathfrak{p}}}^1(\chi_{\mathfrak{p}})(a_{\mathfrak{p}}) = 0$$

for almost all $\mathfrak{p} \in P$.

Consequently, we obtain a pairing

$$\langle \cdot \rangle_K : H^1(K, \mathbf{Q}/\mathbf{Z}) \otimes I_K \rightarrow \mathbf{Q}/\mathbf{Z}, \quad (2)$$

defined by

$$\langle \chi, (a_{\mathfrak{p}})_{\mathfrak{p} \in P} \rangle_K = \sum_{\mathfrak{p} \in P} \langle \chi_{\mathfrak{p}}, a_{\mathfrak{p}} \rangle_{\mathfrak{p}}.$$

Then, we have the following result.

PROPOSITION 3.1. (c.f. [9, p.57, II, Proposition 1.2]) Let k be an n -dimensional local field. Let X be a smooth proper curve over k and let $K = R(X)$. Then, if $a = (a_{\mathfrak{p}})_{\mathfrak{p} \in P}$ is in the diagonal image of $K_{n+1}^M(K)$ in I_K , we have, for any $\chi \in H^1(K, \mathbf{Q}/\mathbf{Z})$,

$$\langle \chi, a \rangle_K = 0.$$

Proof. Let p be a prime number. Then, it is sufficient to prove the statement for an element of $H^1(K, \mathbf{Z}/p\mathbf{Z})$. The proof is similar to [9, p.57, II, Proposition 1.2]. \square

By applying this result, we obtain a generalization of [9, p.76, II, Theorem 7.1].

COROLLARY 3.1. Let L be a cyclic extension of K in which almost all $\mathfrak{p} \in X^{(0)}$ split completely. Then, all $\mathfrak{p} \in X^{(0)}$ split completely in the extension.

Proof. Now, we have a finite morphism $X \rightarrow \mathbf{P}_k^1$. Every finite subset of \mathbf{P}_k^1 is contained in an open affine set, and so is X . Let I be a subset of $X^{(0)}$ such that the elements of $X^{(0)} \setminus I$ split completely in L . Then, there exists an open affine scheme $\text{Spec}(R)$ of X that contains I .

For any positive integer i and $\mathfrak{p} \in P$, let $U^i(K_{\mathfrak{p}}^*) = \text{Ker}(R_{\mathfrak{p}}^* \rightarrow (R_{\mathfrak{p}}/\mathfrak{p}^i R_{\mathfrak{p}})^*)$ and let $U^i K_{n+1}^{n+1}(K_{\mathfrak{p}})$ be the subgroup generated by symbols $\{u, x_1, \dots, x_n\}$ with $u \in U^i(K_{\mathfrak{p}})$ and $x_1, \dots, x_n \in K_{\mathfrak{p}}^*$.

Then, the image of

$$K_{n+1}^M(K) \rightarrow \prod_{\mathfrak{p} \in I} K_{n+1}^M(K_{\mathfrak{p}}) / U^i K_{n+1}^M(K_{\mathfrak{p}})$$

is surjective by the approximation theorem for a Dedekind domain R .

Moreover, a norm subgroup of $K_{n+1}^M(K_{\mathfrak{p}})$ contains $U^i K_{n+1}^M(K_{\mathfrak{p}})$ for a sufficiently large i . Hence, the statement follows from Proposition 3.1 and Theorem 3.2 (i), (ii). \square

Let C_K be the quotient of I_K by the image of the subgroup $K_{n+1}^M(K)$. By Proposition 3.1 and the pairing (2), we obtain a pairing

$$\langle \cdot \rangle_K : H^1(K, \mathbf{Q}/\mathbf{Z}) \otimes C_K \rightarrow \mathbf{Q}/\mathbf{Z}. \quad (3)$$

Now, we observe that the quotient of C_K by the image of the subgroup

$$\prod_{\mathfrak{p} \in P} K_{n+1}^M(R_{\mathfrak{p}})$$

is canonically isomorphic to $SK_n(X)$, because the quotient of $K_{n+1}^M(K_{\mathfrak{p}})$ by $K_{n+1}^M(R_{\mathfrak{p}})$ is isomorphic to $K_n^M(\kappa(\mathfrak{p}))$ via the boundary map $\partial_{\mathfrak{p}}$.

Consequently, the pairing (3) induces the pairing $H^1(K, \mathbf{Q}/\mathbf{Z}) \otimes SK_n(X) \rightarrow \mathbf{Q}/\mathbf{Z}$, and we obtain the homomorphism $\tau : SK_n(X) \rightarrow H^1(K, \mathbf{Q}/\mathbf{Z})^* = \pi_1^{ab}(X)$.

COROLLARY 3.2. Let k be an n -dimensional local field where the characteristic of k is 0 and X is a smooth proper scheme over k . Then, we obtain the homomorphism $\tau : SK_n(X) \rightarrow \pi_1^{ab}(X)$.

Proof. By the same argument as that in the proof of [10, p.259, Lemma (3.2)], it is sufficient to prove the statement for a proper smooth curve over k . Thus, the proof is complete. \square

Then, we have the following result for homomorphisms ∂ and τ .

LEMMA 3.1. (c.f. [10, p.261, Lemma 3.11 (2), (3)]) Let k be an n -dimensional local field. Suppose that X and Y satisfy Assumption 3.3. Then,

- (i). The map ∂ is surjective.
- (ii). We have the commutative diagram

$$\begin{array}{ccc} SK_n(X) & \xrightarrow{\tau} & \pi_1^{ab}(X) \\ \partial \downarrow & & \downarrow \delta \\ SK_{n-1}(Y) & \xrightarrow{\tau} & \pi_1^{ab}(Y). \end{array} \quad (4)$$

Here, δ is the composition map $\pi_1(X) \twoheadrightarrow \pi_1(\mathfrak{X}) \xleftarrow{\sim} \pi_1(Y)$, in which the first map is surjective because X is normal [4, p.41, I, Examples 5.2 (b)] and the second map is an isomorphism by [1, Theorem (3.1) and (3.4)].

Proof. The proof is similar to [10, p.261, Lemma 3.11 (2), (3)]. We use Theorem 3.2 (iii) to prove (ii). \square

The following is known as a generalization of a Henselian regular local ring .

LEMMA 3.2. ([10, p.263, Lemma 3.15]) Let A be a Henselian regular local ring of dimension ≥ 2 , with perfect residue field F and quotient field K . Let T be a regular parameter of A , and suppose that in case $\text{ch}(K) = 0$ and $\text{ch}(K) = p > 0$, (T) is the unique prime ideal of height one that divides (p) . Put $U = \text{Spec}(A[1/T])$.

Then, if $\chi \in H^1(U, \mathbf{Q}/\mathbf{Z})$ induces an unramified character $\chi_u \in H^1(u, \mathbf{Q}/\mathbf{Z})$ for each $u \in U^{(0)}$, χ comes from $H^1(\text{Spec}(A), \mathbf{Q}/\mathbf{Z})$.

We show the following fact by using this result and higher local class field theory.

PROPOSITION 3.2. (c.f. [10, Proposition 3.12]) Let k be an n -dimensional local field. Suppose that \mathfrak{X} , X , and Y satisfy Assumption 3.3. Moreover, \mathfrak{X} is smooth over S . Let $\chi \in H^1(X, \mathbf{Q}/\mathbf{Z})$ and $\tilde{\chi} : \text{SK}_n(X) \rightarrow \mathbf{Q}/\mathbf{Z}$ be the induced homomorphism. Then, the following are equivalent:

- (i). χ comes from the subgroup

$$H^1(Y, \mathbf{Q}/\mathbf{Z}) \simeq H^1(\mathfrak{X}, \mathbf{Q}/\mathbf{Z}) \hookrightarrow H^1(X, \mathbf{Q}/\mathbf{Z}).$$

- (ii). $\tilde{\chi}$ factors through the map ∂ .

Proof. The proof is similar to that of [10, Proposition 3.12], as follows. (i) implies (ii) by Lemma 3.1. Therefore, it is sufficient to show that (ii) implies (i). For $x \in Y^{(0)}$, let A_x be the Henselization of the local ring \mathcal{O}_x of \mathfrak{X} at x . Let $U_x = \text{Spec}(A_x) \times_{\mathfrak{X}} X$, and let $\chi_x \in H^1(U_x, \mathbf{Q}/\mathbf{Z})$ be the restriction of χ . Then, the canonical morphism $U_x \rightarrow X$ induces a bijection

$$j : (U_x)^{(0)} \rightarrow \{u \in X^{(0)} | u \rightarrow x\},$$

and for $u \in (U_x)^{(0)}$, $\kappa(j(u)) \simeq \kappa(u)$.

We assume (ii). Then, for every $u \in (U_x)^{(0)}$, the restriction $\chi_{x,u} \in H^1(u, \mathbf{Q}/\mathbf{Z})$ of χ_x corresponds to an unramified extension of $\kappa(u)$ by Theorem 3.2 (iii). Let t be a prime element of \mathcal{O}_k . Then, t becomes a regular parameter of A_x by the assumption of X . Hence, χ_x comes from $H^1(\text{Spec}(A_x), \mathbf{Q}/\mathbf{Z})$ by Lemma 3.2. Therefore, (ii) implies (i) by descent theory. \square

Definition 3.4. Let Z be a Noetherian scheme. A finite etale covering

$f : U \rightarrow Z$ is called a *c.s. covering* if any closed point x of Z *splits completely* in the covering, that is, $\text{Spec } \kappa(x) \times_Z U$ is isomorphic to a finite sum $\text{Spec } \kappa(x)$, where $\kappa(x)$ is the residue field of x . Let $\pi_1^{c.s.}(Z)$ denote the quotient group of $\pi_1^{ab}(Z)$ that classifies c.s. coverings of Z .

Let X be a normal scheme. Then,

$$\text{Hom}_{cont}(\pi_1^{c.s.}(X), \mathbf{Q}/\mathbf{Z}) = \text{Ker} \left(H^1(X, \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{\mathfrak{p} \in X^{(0)}} H^1(\kappa(\mathfrak{p}), \mathbf{Q}/\mathbf{Z}) \right)$$

by Definition 3.4. In addition,

$$\begin{aligned} & \text{Ker} \left(H^1(X, \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{\mathfrak{p} \in X_{(1)}} H^1(\kappa(\mathfrak{p}), \mathbf{Q}/\mathbf{Z}) \right) \\ &= \text{Ker} \left(H^1(R(X), \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{\mathfrak{p} \in X_{(1)}} H^1(\widetilde{R(X)}_{\mathfrak{p}}, \mathbf{Q}/\mathbf{Z}) \right). \end{aligned}$$

LEMMA 3.3. (c.f. [5, p.524, Lemma 5.4]) Let k be an n -dimensional local field. Suppose that \mathfrak{X} , X , and Y satisfy Assumption 3.3. Moreover, \mathfrak{X} is smooth over S . Then, the specialization map $\delta_{\mathfrak{X}}$ (where we omitted suitable base points) is surjective and induces an isomorphism

$$\pi_1(X)^{c.s.} \simeq \pi_1(Y)^{c.s.}. \quad (5)$$

Proof. The proof is similar to [5, p.524, Lemma 5.4]. The surjectivity of δ_X follows from the definition. Therefore, the surjectivity of (5) follows. The injectivity of (5) follows from Proposition 3.2. \square

Let $k_0, k_1, \dots, k_n = K$ be a sequence of fields as in Definition 3.1, and let X be a proper algebraic curve over K . Let \mathcal{O}_{k_i} ($0 \leq i \leq n$) be a Henselian discrete valuation with quotient field k_i . When X satisfies the following condition, we say that X has a good reduction in each step.

X has a good reduction, that is, there is a proper smooth morphism $\mathfrak{X} \rightarrow \text{Spec } \mathcal{O}_{k_n}$. For any irreducible component X_{n-1} of $\mathfrak{X} \otimes_{\mathcal{O}_{k_n}} k_{n-1}$, it has a good reduction. Moreover, we can repeat the above step until we obtain a proper smooth morphism $X_0 \rightarrow \text{Spec } k_0$.

Then, we have the following result.

PROPOSITION 3.3. Let k be an n -dimensional local field. Let X be a proper algebraic curve over k and regular with a good reduction in each step, and let P be the set that contains almost all closed points of X . Then,

$$\text{Ker} \left(H^1(R(X), \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{\mathfrak{p} \in P} H^1(\widetilde{R(X)}_{\mathfrak{p}}, \mathbf{Q}/\mathbf{Z}) \right) = 0.$$

Proof. Proposition 3.3 follows from Lemma 3.3 and Corollary 3.1. \square

4 Proof of Theorem 4.1

First, we prove the following Lemmas in order to prove the main result.

LEMMA 4.1. Let A be a Dedekind domain; k , its quotient field; and X , a proper algebraic curve over k . Let $k_{\mathfrak{p}}$ be the Henselization of k at $\mathfrak{p} \in \text{Spec}(A)$. Suppose that the characteristic of k is 0.

Then, $X \times_k k_{\mathfrak{p}}$ has a good reduction for almost all $\mathfrak{p} \in \text{Spec}(A)$.

Proof. Let L be the function field of X . Let Y be the normalization of \mathbf{P}_A^1 in L . Then, $Y \rightarrow \mathbf{P}_A^1$ is finite ([4, I, Proposition 1.1]). Hence, $Y \rightarrow \mathbf{P}_A^1$ is proper ([4, I, Proposition 1.4]). Therefore, $f : Y \rightarrow \operatorname{Spec} A$ is proper, and so is $f' : Y \times_{\operatorname{Spec} A} \operatorname{Spec} k \rightarrow \operatorname{Spec} k$, which is the base change of f by $\operatorname{Spec} k \rightarrow \operatorname{Spec} A$. Since $Y \times_{\operatorname{Spec} A} \operatorname{Spec} k$ is regular, f' is smooth and $Y \times_{\operatorname{Spec} A} \operatorname{Spec} k \simeq X$.

Let T be the set of elements in Y that is smooth over $\operatorname{Spec} A$. Then, the proper map f takes the complement of T to a closed set B of $\operatorname{Spec} A$. Since f' is smooth, B is a proper subset of $\operatorname{Spec} A$, and hence, it consists of finite elements.

Let $A_{\mathfrak{p}}$ be the localization of A at $\mathfrak{p} \in \operatorname{Spec} A \setminus B$. Then, $Y \times_{\operatorname{Spec} A} \operatorname{Spec} A_{\mathfrak{p}} \rightarrow \operatorname{Spec} A_{\mathfrak{p}}$, which is the base change of f by $\operatorname{Spec} A_{\mathfrak{p}} \rightarrow \operatorname{Spec} A$, is proper and smooth. Therefore, the statement follows. \square

LEMMA 4.2. Let k be an arbitrary field with notations A, X as in Lemma 4.1. Let P be the set that contains almost all closed points of X . Let $\operatorname{kg}l(P)$ denote the kernel of the local-global map

$$\operatorname{H}^1(R(X), \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{x \in P} \operatorname{H}^1(\widetilde{R(X)}_x, \mathbf{Q}/\mathbf{Z}).$$

Suppose that there exists a separable algebraic field extension k_i over k for $i \in I$ and that the natural homomorphism

$$\operatorname{H}^1(k, \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{i \in I} \operatorname{H}^1(k_i, \mathbf{Q}/\mathbf{Z})$$

is injective. Let P_i be the inverse image of P by $X \times_k k_i \rightarrow X$.

Then, P_i contains almost all closed points of $X \otimes_k k_i$ and the homomorphism

$$\operatorname{kg}l(P) \rightarrow \prod_{i \in I} \operatorname{kg}l(P_i) \tag{6}$$

is injective.

Proof. Let K be a field; K_s , its separable closure; and Z , an algebraic curve over K . Then, $Z \times_K K_s \rightarrow Z$ is a Galois covering, and its Galois group is $G(K_s/K)$. Then, there exists a Hochschild-Serre spectral sequence

$$\operatorname{H}^p(G(K_s/K), \operatorname{H}^q(Z \times_K K_s, \mathbf{Q}/\mathbf{Z})) \Rightarrow \operatorname{H}^{p+q}(Z, \mathbf{Q}/\mathbf{Z})$$

(cf, [4, III, Theorem 2.20 and Remark 2.21]), and hence, we have an exact sequence

$$0 \rightarrow \operatorname{H}^1(K, \mathbf{Q}/\mathbf{Z}) \rightarrow \operatorname{H}^1(Z, \mathbf{Q}/\mathbf{Z}) \rightarrow \operatorname{H}^1(Z \times_K K_s, \mathbf{Q}/\mathbf{Z})^{G(K_s/K)}.$$

Therefore, the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{H}^1(k, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & \operatorname{H}^1(X, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & \operatorname{H}^1(X \times_k k_s, \mathbf{Q}/\mathbf{Z})^{G(k_s/k)} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_{i \in I} \operatorname{H}^1(k_i, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & \prod_{i \in I} \operatorname{H}^1(X \times_k k_i, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & \prod_{i \in I} \operatorname{H}^1(X \times_k k_s, \mathbf{Q}/\mathbf{Z})^{G(k_s/k_i)} \end{array}$$

is commutative. Then, the vertical map on the right-hand side is clearly injective, and so is that on the left-hand side by the assumption; hence, that of the middle,

$$H^1(X, \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{i \in I} H^1(X \times_k k_i, \mathbf{Q}/\mathbf{Z}), \quad (7)$$

is injective. Since the homomorphism (6) is obtained by the commutative diagram

$$\begin{array}{ccc} H^1(X, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^1(X \times_k k_i, \mathbf{Q}/\mathbf{Z}) \\ \downarrow & & \downarrow \\ \prod_{y \in P} H^1(\kappa(y), \mathbf{Q}/\mathbf{Z}) & \longrightarrow & \prod_{x \in P_i} H^1(\kappa(x), \mathbf{Q}/\mathbf{Z}), \end{array}$$

the homomorphism (6) is injective. \square

Let F be a finitely generated field over \mathbf{Q} . Let the Kronecker dimension of F be n . Then, there is a finitely generated field F' such that F' is algebraic closed in F and the Kronecker dimension of F' is $n - 1$.

Moreover, we have the regular and proper algebraic curve X over F' such that $R(X) = F$. Hence, there exists an inclusion map $F \rightarrow F_{n,i}$, where $F_{n,i}$ is an n -dimensional local field with a sequence of unramified extension fields $k_0, \dots, k_n = F_{n,i}$ satisfying the conditions in Definition 3.1.

Then, the following main result follows from the above lemmas.

THEOREM 4.1. (i). Let k be a finitely generated field over \mathbf{Q} . Let X be a regular and proper algebraic curve over k . Let P be the set that consists of almost all closed points of X .

Then, the local-global map of the character group

$$H^1(R(X), \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{p \in P} H^1(\widetilde{R(X)}_p, \mathbf{Q}/\mathbf{Z}) \quad (8)$$

is injective. Hence, the homomorphism (8) is injective, regardless of whether X is complete.

(ii). Moreover, let the Kronecker dimension of k be n . Suppose that the set $\{k_{n,i}\}_{i \in P_n}$ is such that an n -dimensional local field $k_{n,i}$ is derived from k as shown above and $X \times_k k_{n,i}$ has a good reduction in each step. Then, the local-global map

$$H^1(k, \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{i \in P_n} H^1(k_{n,i}, \mathbf{Q}/\mathbf{Z}) \quad (9)$$

is injective. Moreover,

$$H^1(R(X), \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{i \in P_n} H^1(R(X \times_k k_{n,i}), \mathbf{Q}/\mathbf{Z}) \quad (10)$$

is injective.

Proof. We prove the statement by induction. Suppose that k is an algebraic number field and X is the ring of integers of k . Then, the homomorphism (8) is injective by [2, p.180, 8.8 Corollary]. When (i) holds for k_n with the Kronecker dimension $n \geq 1$, that is, the homomorphism (8) is injective, we see that (9) is injective by Lemma 4.1. Let Z be a regular scheme. Then,

$$0 \rightarrow H^1(Z, \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(R(Z), \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{\mathfrak{p} \in Z_{(1)}} H^1(R(\text{Spec}(\mathcal{O}_{Z, \mathfrak{p}})), \mathbf{Q}/\mathbf{Z}) \quad (11)$$

is an exact sequence. Since the sequence (11) is exact and the homomorphism (7) in the proof of Lemma 4.2 is injective, we see that the homomorphism (10) is injective, that is, (ii) holds.

When (ii) holds for k_n , we see that (i) holds for k_n as follows. Let $\{\tilde{k}_i\}_{i \in I}$ be the set of n -dimensional local fields that satisfy the assumption of (ii). Then, $\{\tilde{k}_i\}_{i \in I}$ satisfy the assumption of Lemma 4.2 by (9). Hence, it is sufficient to show that $\text{kgl}(P_i) = 0$ to show (i). It follows from Proposition 3.3. Therefore, the proof is complete. \square

REMARK 4.1. A method of algebraic geometry enables us to prove Theorem 4.1 by induction. Theorem 4.1 follows from the Hasse principle for the character group of a global field, and it is proved by an algebraic method (c.f. [2, p.180, 8.8 Corollary]). Therefore, Theorem 4.1 is proved by an algebraic method.

By applying the main result, we obtain the following result.

COROLLARY 4.1. Let \mathcal{X} be a connected normal scheme of finite type over $\text{Spec}(\mathbf{Z})$ where the characteristic of \mathcal{X} is 0. Suppose that $i \leq \dim(\mathcal{X})$. Then,

$$H^1(\mathcal{X}, \mathbf{Q}/\mathbf{Z}) \rightarrow \prod_{\mathfrak{p} \in \mathcal{X}_{(i)}} H^1(\kappa(\mathfrak{p}), \mathbf{Q}/\mathbf{Z}) \quad (12)$$

is injective. The statement for $i = \dim(\mathcal{X})$ is equivalent to the following:

If $\mathcal{Y} \rightarrow \mathcal{X}$ is a connected c.s. covering, $\mathcal{Y} \rightarrow \mathcal{X}$ is an isomorphism.

Proof. Let $\text{Spec}(A)$ be an open affine scheme of \mathcal{X} . Let $f : \mathbf{Z} \rightarrow A$ be a ring homomorphism that corresponds to a morphism of schemes

$$\text{Spec}(A) \subset \mathcal{X} \rightarrow \text{Spec}(\mathbf{Z}).$$

Suppose that $R(\mathcal{X})$ is an algebraic function field in one variable over K . Then, we have a normal ring B with $R(B) = K$ and $g : B \rightarrow A$, which is an extension of f . Since $\text{Spec}(A) \otimes_{\text{Spec}(B)} K$ is normal, Theorem 4.1 holds in this case, and the homomorphism (12) is injective for $i = 1$.

Moreover, let $A^{\mathfrak{p}}$ be the normalization of A/\mathfrak{p} . Since $\text{Spec}(A^{\mathfrak{p}}) \rightarrow \text{Spec}(A/\mathfrak{p})$ is finite by [3, Proposition (7.8.6)], $A^{\mathfrak{p}}$ is a normal scheme of finite type over \mathbf{Z} . In addition, the homomorphism $H^1(\mathcal{X}, \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(\kappa(\mathfrak{p}), \mathbf{Q}/\mathbf{Z})$ goes through $H^1(\text{Spec } A^{\mathfrak{p}}, \mathbf{Q}/\mathbf{Z})$, and $H^1(\text{Spec } A^{\mathfrak{p}}, \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(\kappa(\mathfrak{p}), \mathbf{Q}/\mathbf{Z})$ is injective. Now, Corollary 4.1 holds for $\text{Spec } A^{\mathfrak{p}}$ and $i = 1$. Therefore, it holds for \mathcal{X} and $i = 2$. By repeating the above argument, we can see that the statement holds for any $i \leq \dim(\mathcal{X})$. \square

References

- [1] M. ARTIN, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math., **36**, (1969), 23–58.
- [2] J.W.S. CASSELS AND A. FRÖHLICH, *Algebraic number theory*, Academic Press, London, New York, 1967.
- [3] A. GROTHENDIECK, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965).
- [4] J. MILNE, *Étale Cohomology*, Princeton Univ. Press, Princeton, 1980.
- [5] U. JANNSEN AND S. SAITO, *Kato homology of arithmetic schemes and higher class field theory over local fields*, Doc. Math. 2003, Extra Vol., 479–538 (electronic).
- [6] K. KATO, *A generalization of local class field theory by using K -groups, II*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 603–683.
- [7] K. KATO, *Milnor K -theory and the Chow group of zero cycles* (Boulder, CO, 1983), Contemp. Math. **55**, Amer. Math. Soc., Providence, 1986, pp. 241–253 .
- [8] K. KATO, Existence theorem for higher local fields, in *Invitation to Higher Local Fields*, eds. I. Fesenko and M. Kurihara, Geometry and Topology Monographs, Vol. 3 (University of Warwick, 2000), pp. 165–195.
- [9] S. SAITO, *Class field theory for curves over local fields*, Journal of Number Theory, (1) **21** (1985), 44–80.
- [10] S. SAITO, *Unramified class field theory of arithmetical schemes*, Ann. of Math. (2) **121**(1985), 251–281.